Problem set: Green functions

6.1. Green functions and the q-propagator

(a) Show that the retarded single particle Green function, \( G^R \), defined by the relation

\[
[i\hbar \partial_t - H_0(r) - V(r)] G^R(r, t; r', t') = \delta(r - r') \delta(t - t') \quad (1)
\]

is a propagator, i.e.,

\[
\psi(r, t) = \int \! dr' \, G(r, t; r', t') \psi(r', t') . \quad (2)
\]

Hint: Substitute Eq. (2) into the Schrödinger equation

\[
[i\hbar \partial_t - H_0(r) - V(r)] \psi(r, t) = 0 \quad (3)
\]

and use definition (1). Please note that you should distinguish between forward and backward propagation. Consider, e.g., the former and use the retarded propagator defined as

\[
\vartheta(t - t') \psi(r, t) = i\hbar \int \! dr' \, G^R(r, t; r', t') \psi(r', t') . \quad (4)
\]

(b) Show that the retarded propagator

\[
G^R(r, t; r', t') = \frac{1}{i\hbar} \vartheta(t - t') \langle r \mid e^{\frac{i}{\hbar}H(t-t')} \mid r' \rangle \quad (5)
\]

is a solution of the differential equation (1).

Hint: Differentiate Eq. (1) w.r.t. time using the fact that the derivative of the Heaviside theta function is a delta function, and that

\[
\langle r \mid H \mid \phi \rangle = H(r) \langle r \mid \phi \rangle
\]
6.2. **The Lehmann representation**

Consider the retarded Green function in the so-called Lehmann representation \( \{ |n\rangle \} \) of the set of eigenstates of the full Hamiltonian \( H \).

(a) Show that in this basis the Fourier transform w.r.t. time of

\[
G^R(\nu, t) = \frac{1}{i\hbar} \vartheta(t) \left\langle \left[ c(t), c_\nu^\dagger(0) \right]_+ \right\rangle
\]

is

\[
G^R(\nu, \omega) = \frac{1}{Z} \sum_{n,n'} \frac{\langle n|c_\nu^\dagger|n' \rangle \langle n'|c_\nu^\dagger|n \rangle}{\hbar \omega + E_n - E_{n'} + i0^+} (e^{-\beta E_n} + e^{-\beta E_{n'}}).
\]

(b) Show that \( A(\nu, \omega) := -(1/\pi) \text{Im} G^R(\nu, \omega) \) is always positive, and that is frequency-normalized, i.e.,

\[
\int d(\hbar \omega) A(\nu, \omega) = 1.
\]

**Note:** The function \( A \) is called spectral density or, in other contexts density of states (DOS); it yields a complete information about the system. In fact due to the analyticity properties of the Green function the real part of \( G^R \) can be obtained from \( A \) via a Hilbert transformation (in physics more known as Kramers-Kronig relation). It is worth to note that \( A \) contains information on the excitation spectrum on the system\(^5\).

---