Problem set: Second quantization I (bosonic gymnastic)

3.1. Bosonic commutation relations

Refresh the physics of the simple harmonic oscillator

\[ H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}, \]

which can be written in “second quantized” form, by expressing \(x\) and \(p\) in terms of boson creation and annihilation operators:

\[ H = \hbar \omega \left( a^\dagger a + \frac{1}{2} \right), \quad a^\dagger = \frac{1}{\sqrt{2\hbar}} \left( \sqrt{m\omega} \hat{x} - i \frac{\hat{p}}{\sqrt{m\omega}} \right). \]

(a) Show that the following basis commutation relations hold

\[ [a, a^\dagger] = 1, \quad [a, a] = 0, \quad a |0\rangle = 0 \]

where \([A, B] = AB - BA, |0\rangle\) the vacuum, and \(\dagger\) indicates the Hilbert space adjoint. From these, determine all normalized eigenstates \(|n\rangle\) \((|n, m\rangle = \delta_{nm})\) of \(a^\dagger a\), and show that they have the following properties,

\[ a^\dagger a |n\rangle = n |n\rangle, \quad n = 0, 1, 2, \ldots \]
\[ a |n\rangle = \sqrt{n} |n - 1\rangle, \]
\[ a^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle. \]

(b) Compute \(F = -k_B T \log Z\) with

\[ Z = \text{Tr} \left\{ \exp \left[ -\frac{\hbar \omega}{k_B T} \left( a^\dagger a + \frac{1}{2} \right) \right] \right\} = \sum_n \langle n, \exp \left[ -\frac{\hbar \omega}{k_B T} \left( a^\dagger a + \frac{1}{2} \right) \right] n \rangle \]

3.2. Calculating with bosonic operators I

(a) Show that for two non commuting bosonic operators \(A\), and \(B\) it holds

\[ [A, B^n] = \sum_{k=0}^{n-1} B^k [A, B] B^{n-1-k}. \]
(b) Prove —using (a)— that for bosonic operators $b$, $b^\dagger$

\[
[b, (b^\dagger)^n] = n(b^\dagger)^{n-1} = \frac{\partial(b^\dagger)^n}{\partial b^\dagger},
\]

\[
[b^\dagger, b^n] = -nb^{n-1} = -\frac{\partial b^n}{\partial b},
\]

\[
[b, f(b^\dagger)] = \frac{\partial f(b^\dagger)}{\partial b^\dagger},
\]

\[
[b^\dagger, f(b)] = -\frac{\partial f(b)}{\partial b},
\]

where the functions $f(b^\dagger)$ and $f(b)$ are representable as a power series ($[b, b^\dagger] = 1$).

3.3. Calculating with bosonic operators II (optional)

(a) Using the previous arguments (Ex 3.3) show that the following relations hold

\[
g_1(\alpha; b, b^\dagger) = e^{-ab^\dagger}be^{ab^\dagger} = b + \alpha, \quad h_1(\alpha; b, b^\dagger) = e^{-ab^\dagger}e^{ab^\dagger} = b^\dagger - \alpha.
\]

In a similar fashion, simplify the following expressions

\[
g_2(\alpha; b, b^\dagger) = e^{-(\alpha + ab^\dagger - ab)b^\dagger}e^{(\alpha + ab^\dagger - ab)}, \quad h_2(\alpha; b, b^\dagger) = e^{-(\alpha + ab^\dagger - ab)b^\dagger}e^{(\alpha + ab^\dagger - ab)},
\]

\[
g_3(\alpha; b, b^\dagger) = e^{-ab^\dagger}be^{ab^\dagger}b, \quad h_3(\alpha; b, b^\dagger) = e^{-ab^\dagger}b^\dagger e^{ab^\dagger}b.
\]

**Hint:** Introduce a dummy variable $\lambda$ as in

\[
g_i(\lambda, \alpha; b, b^\dagger) = e^{-\lambda f(\alpha; b, b^\dagger)}b e^{\lambda f(\alpha; b, b^\dagger)}
\]

and calculate the derivative $\partial g_i(\lambda, \alpha; b, b^\dagger)/\partial \lambda$. Same thing for $h_i(\lambda, \alpha; b, b^\dagger)$.

(b) Deduce the results in (a) from the differential equations for $g_i$ and $h_i$, obtained by differentiating $g_i$ and $h_i$ with respect to $\alpha$.

(c) Prove the relation

\[
e^{-ab^\dagger}e^{\beta b}e^{ab^\dagger} = e^{\beta \alpha}e^{\beta b}
\]

(d) Prove the identity

\[
e^{ab^\dagger}e^{\beta b} = \exp\left(\alpha b^\dagger + \beta b - \frac{\alpha \beta}{2}\right)
\]

**Hint:** Show first that the quantity

\[
f(\lambda) = e^{\lambda ab^\dagger}e^{\lambda \beta b}e^{\lambda^2 \frac{ab^\dagger}{2}}
\]

satisfies the following relation

\[
\frac{\partial f(\lambda)}{\partial \lambda} = (\alpha b^\dagger + \beta b) f(\lambda).
\]